Exercise 4

A particle moves according to a law of motion $s = f(t)$, $t \geq 0$, where t is measured in seconds and s in feet.

(a) Find the velocity at time t.

(b) What is the velocity after 1 second?

(c) When is the particle at rest?

(d) When is the particle moving in the positive direction?

(e) Find the total distance traveled during the first 6 seconds.

(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.

(g) Find the acceleration at time t and after 1 second.

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 6$.

(i) When is the particle speeding up? When is it slowing down?

$$f(t) = t^2 e^{-t}$$

Solution

Part (a)

To find the velocity, take the derivative of the position function.

$$\begin{aligned}
v(t) &= \frac{ds}{dt} \\
&= \frac{d}{dt}(t^2 e^{-t}) \\
&= \left[\frac{d}{dt}(t^2)\right] e^{-t} + t^2 \left[\frac{d}{dt}(e^{-t})\right] \\
&= (2t)e^{-t} + t^2 \left[e^{-t} \cdot \frac{d}{dt}(-t)\right] \\
&= 2te^{-t} + t^2[e^{-t} \cdot (-1)] \\
&= te^{-t}(2 - t)
\end{aligned}$$

Part (b)

The velocity after 1 second has elapsed is

$$v(1) = (1)e^{-1}(2 - 1) = \frac{1}{e} \frac{\text{feet}}{\text{second}}.$$

Part (c)

To find when the particle is at rest, set the velocity function equal to zero and solve the equation for t.

$$v(t) = 0$$

$$te^{-t}(2-t) = 0$$

$$t(2-t) = 0$$

$$t = 0 \quad \text{or} \quad 2 - t = 0$$

$$t = \{0, 2\}$$

The particle is at rest when $t = 0$ and $t = 2$.

Part (d)

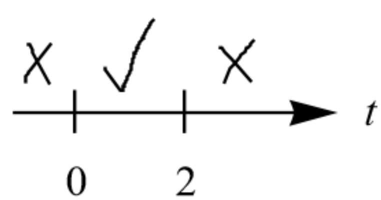
To find when the particle is moving in the positive direction, find what values of t satisfy $v(t) > 0$.

$$v(t) > 0$$

$$te^{-t}(2-t) > 0$$

$$t(2-t) > 0$$

The critical values are 0 and 2. Partition the number line at these points and test whether the inequality is true within each interval.



Therefore, the particle is moving in the positive direction for $0 < t < 2$.

Part (e)

The distance travelled in $0 \le t < 2$ is

$$|s(2) - s(0)| = \left|(2)^2 e^{-2} - (0)^2 e^{-0}\right| = \frac{4}{e^2},$$

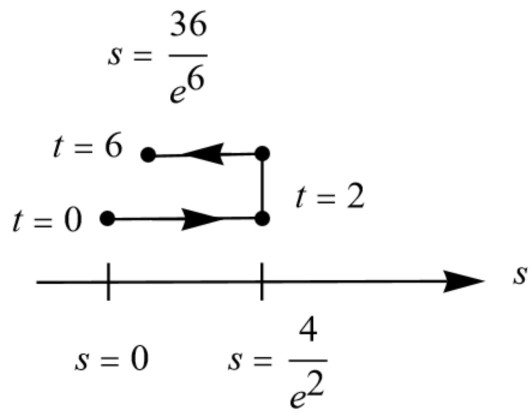
and the distance travelled in $2 < t \le 6$ is

$$|s(6) - s(2)| = \left|(6)^2 e^{-6} - (2)^2 e^{-2}\right| = \frac{4}{e^2} - \frac{36}{e^6}.$$

Consequently, the total distance travelled in $0 \leq t \leq 6$ is $\frac{4}{e^2} + \frac{4}{e^2} - \frac{36}{e^6} = \frac{8}{e^2} - \frac{36}{e^6}$ feet.

Part (f)

Below is an illustration of the particle's motion from $t = 0$ to $t = 6$.



Part (g)

Calculate the derivative of the velocity to get the acceleration.

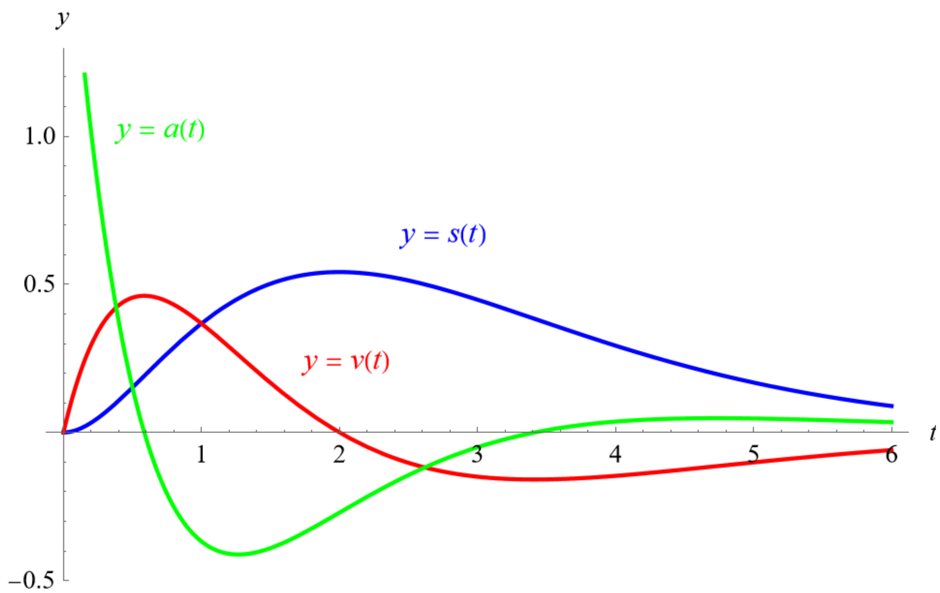
$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= \frac{d}{dt}[te^{-t}(2-t)] \\ &= \frac{d}{dt}[e^{-t}(2t-t^2)] \\ &= \left[\frac{d}{dt}(e^{-t})\right](2t-t^2) + e^{-t}\left[\frac{d}{dt}(2t-t^2)\right] \\ &= \left[e^{-t} \cdot \frac{d}{dt}(-t)\right](2t-t^2) + e^{-t}(2-2t) \\ &= [e^{-t} \cdot (-1)](2t-t^2) + e^{-t}(2-2t) \\ &= e^{-t}(t^2-4t+2) \end{aligned}$$

The acceleration after 1 second is

$$a(1) = e^{-1}[1^2 - 4(1) + 2] = -\frac{1}{e} \frac{\text{feet}}{\text{second}^2}.$$

Part (h)

Below is a plot of the position, velocity, and acceleration versus time for $0 \leq t \leq 6$.



Part (i)

The particle is speeding up when

$$e^{-t}(t^2 - 4t + 2) > 0$$

$$t^2 - 4t + 2 > 0. \tag{1}$$

In order to factor this polynomial, solve the corresponding equation:

$$t^2 - 4t + 2 = 0$$

$$t = \frac{4 \pm \sqrt{16 - 4(2)}}{2}$$

$$t = \frac{4 \pm \sqrt{8}}{2}$$

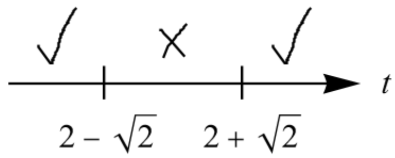
$$t = 2 \pm \sqrt{2}.$$

As a result, (1) becomes

$$[t - (2 + \sqrt{2})][t - (2 - \sqrt{2})] > 0.$$

The critical values are $2 - \sqrt{2}$ and $2 + \sqrt{2}$.

Partition the number line at these points and test whether the inequality is true within each interval.



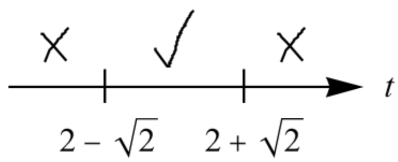
Consequently, the particle is speeding up when $0 < t < 2 - \sqrt{2}$ or $2 + \sqrt{2} < t < 6$. The particle is slowing down when

$$e^{-t}(t^2 - 4t + 2) < 0$$

$$t^2 - 4t + 2 < 0$$

$$[t - (2 + \sqrt{2})][t - (2 - \sqrt{2})] < 0$$

The critical values are $2 - \sqrt{2}$ and $2 + \sqrt{2}$. Partition the number line at these points and test whether the inequality is true within each interval.



Consequently, the particle is slowing down when $2 - \sqrt{2} < t < 2 + \sqrt{2}$.